

Calabi-Yau duals of torus orientifolds

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ABSTRACT: We study a duality that relates the T^6/\mathbb{Z}_2 orientifold with $\mathcal{N} = 2$ flux to standard fluxless Calabi-Yau compactifications of type IIA string theory. Using the duality map, we show that the Calabi-Yau manifolds that arise are abelian surface (T^4) fibrations over \mathbb{P}^1 . We compute a variety of properties of these threefolds, including Hodge numbers, intersection numbers, discrete isometries, and $H_1(X, \mathbb{Z})$. In addition, we show that S-duality in the orientifold description becomes T-duality of the abelian surface fibers in the dual Calabi-Yau description. The analysis is facilitated by the existence of an explicit Calabi-Yau metric on an open subset of the geometry that becomes an arbitrarily good approximation to the actual metric (at most points) in the limit that the fiber is much smaller than the base.

KEYWORDS: Flux compactifications, Superstring Vacua, String Duality.

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1. Introduction

In this investigation, we study a duality that relates the simplest $\mathcal{N} = 2$ warped compactifications to standard type IIA Calabi-Yau compactifications with no flux. We view this investigation as a first step toward the larger goal of understanding which warped compactifications represent new string vacua, and which are just alternative descriptions of more conventional fluxless compactifications.

Warped compactifications, specifically those of D3/D7-type, are an exciting arena in which we have begun to address several important problems in string theory and its applications to cosmology and particle physics. One of the insights of this class of compactifications is in understanding moduli stabilization. Turning on internal Neveu-Schwarz (NS) and Ramond-Ramond (RR) 3-form flux generates a perturbative superpotential that generically stabilizes all complex structure moduli and the axion-dilaton [1–5]. By combining this observation with mechanisms for Kähler moduli stabilization and supersymmetry breaking, this class has been used to construct metastable de Sitter vacua and a variety realizations of inflation in string theory [6, 7]. It is also the ensemble within which statements have been made about the “landscape” of string theory vacua [8]. Most recently, D3/D7-type warped compactifications have been used in MSSM-like model building, where they provide an underlying theory of soft masses that shows promise for explaining some of the usual assumptions on soft masses and the μ -term needed to avoid contradiction with experiment [9]. Finally, warped compactifications come with a holographic interpretation (due to the warping) that often lets us view field theory phenomena in an intuitive geometrical way [10]. For example, the scale of gaugino condensation in the model of Klebanov and Strassler is related to the size of a minimal 3-sphere in the warped deformed conifold [11].

These D3/D7-type warped compactifications, with all of their desirable features, are often presented in a way that suggests they are simply a convenient alternative description of more conventional heterotic or type II compactifications rather than independent vacua. That is, they make certain phenomena manifest that are also present in more complicated ways in conventional type II or heterotic duals. However, it is not clear in which cases there actually exist conventional duals.

As usual, the question is easiest to answer in the case of $\mathcal{N} = 4$ supersymmetry. In this case, the only warped compactification with D3 branes is the T^6/\mathbb{Z}_2 orientifold with 16 D3 branes and 64 O3 planes. This background is dual to type I on T^6 (via six T-dualities), and then to the heterotic string on T^6 (via S-duality), and to type IIA on $K3 \times T^2$ (via heterotic/IIA duality). The $\mathcal{N} = 1$ case is more difficult and we will have nothing to say about it here. On the other hand, the $\mathcal{N} = 2$ case is nontrivial enough to be interesting, yet simple enough to be tractable via standard string dualities. We will focus on the $\mathcal{N} = 2$ case in this investigation.

For $\mathcal{N} = 2$, there exist large web(s) of connected Calabi-Yau vacua. One of the goals in asking about Calabi-Yau dual descriptions of $\mathcal{N} = 2$ orientifolds is to improve our understanding of the connectedness of string theory vacua — a likely prerequisite for understanding potential mechanisms for vacuum selection in string theory. The strongest notion of vacuum connectedness is connectedness in moduli space. In this sense, all Calabi-

Yau manifolds that are hypersurfaces in 4D toric varieties are connected [12], and there are expected to be other large webs of Calabi-Yau vacua that are connected in the same way through flops and conifold transitions [13]. Weaker notions of connectedness also exist and could be relevant in cosmology, which probes more than just the minima of potentials. For example, all T^6/\mathbb{Z}_2 orientifold vacua have been shown to be connected through time-dependent bubbles, including pairs of vacua that preserve unequal amounts of supersymmetry [14]. By relating Calabi-Yau vacua to $\mathcal{N} = 2$ vacua of the T^6/\mathbb{Z}_2 orientifold, we open the possibility of unifying into one large class the two types of connected vacua just described.

An outline of the paper is as follows. In section 2 we review the T^6/\mathbb{Z}_2 orientifold [2, 3] with $\mathcal{N} = 2$ flux. The space of warped compactifications with compact internal manifold that preserve $\mathcal{N} = 2$ supersymmetry includes the T^6/\mathbb{Z}_2 orientifold and the $T^2/\mathbb{Z}_2 \times$ K3 orientifold [4]. We restrict to T^6/\mathbb{Z}_2 for simplicity. After introducing the necessary background on T^6/\mathbb{Z}_2 in section 2.1, we derive the metric moduli constraints resulting from $\mathcal{N} = 2$ flux in section 2.2, and the complete massless spectrum in section 2.3.

In section 3, we describe the chain of dualities that we will use to relate T^6/\mathbb{Z}_2 with $\mathcal{N} = 2$ flux to a purely geometrical type IIA Calabi-Yau compactification — three T-dualities followed by a 9-10 circle swap. This duality was first sketched in ref. [15]. The remainder of section 3 is devoted to implementing the T-dualities. In section 3.1, we review the action of T-duality on NS flux. Then, in section 3.2, we perform the desired three T-dualities to obtain the O6/D6 background dual to T^6/\mathbb{Z}_2 .

In section 4, we perform a lift to M theory on the x^{10} circle and a compactification to IIA on the x^9 circle. We begin in section 4.1 with a qualitative overview of how the chain of dualities results in a Calabi-Yau compactification. Approximate metrics for the dual Calabi-Yau manifolds are derived from the duality chain in section 4.2. In section 4.3, we obtain the Kähler form and (3,0)-form in these approximate metrics and show that they are closed. Section 4.4 provides a further discussion of the complex structure moduli and an enumeration of the Kähler and complex structure moduli in the approximate description.

The purpose of section 5 is to establish an intuition for the sense in which the leading order approximate M theory lift encodes most of the topological data that one would ever want to extract from the complete lift. This section is largely a review of Sen's treatment in ref. [16]. We first explain why the leading order lift is only an approximation to the exact lift whenever orientifold planes are present in the IIA background. We then present examples of increasing complexity, explaining how configurations of D6 branes and O6 planes relate to the homology cycles and harmonic forms in the M theory lift.

We get to the heart of the paper in section 6. Using the leading order description of the dual Calabi-Yau manifolds, we compute a basis of harmonic representatives of $H^2(X_6, \mathbb{Z})$ in section 6.1. We then use this basis to compute the intersection numbers. In section 6.2, we demonstrate that these Calabi-Yau manifolds are abelian surface fibrations over \mathbb{P}^1 . Having identified the fibration structure, we discuss the homology duals to the cohomology basis in section 6.3. Section 6.4 contains a partially heuristic check of the fibration result using a theorem by Oguiso and a feature of the topological amplitude F_1 that we deduce from the T^6/\mathbb{Z}_2 dual description.

In sections 7 and 8, we further sharpen our understanding of the Calabi-Yau geometry. In section 7, we identify discrete gauge symmetries of the T^6/\mathbb{Z}_2 orientifold that arise from $U(1)$ gauge symmetries that are incompletely Higgsed by non-minimally-coupled scalars. These gauge symmetries tell us about torsion cycles and discrete isometries in the dual Calabi-Yau manifolds. In section 8, we identify S-duality of the T^6/\mathbb{Z}_2 orientifold with T-duality of the abelian surface fibers in the Calabi-Yau dual description.

Finally, in section 9, we report attempts to identify some of the dual Calabi-Yau manifolds in terms of known constructions. From a duality based computation of the second Chern class of these Calabi-Yau manifolds, we rule out the possibility that they are quotients of T^6 . This implies one of two possible interesting results, one mathematical and one physical, depending upon whether the Calabi-Yau duals contain rational curves. If they do, then we infer the existence of nonperturbative corrections in the T^6/\mathbb{Z}_2 orientifold that have not been previously deduced by other means. If they do not, then we have mathematically interesting examples of Calabi-Yau manifolds other than quotients of abelian threefolds that do not contain rational curves.

In section 10, we conclude.

2. The T^6/\mathbb{Z}_2 orientifold with $\mathcal{N} = 2$ flux

2.1 Review of T^6/\mathbb{Z}_2

The T^6/\mathbb{Z}_2 orientifold is defined by compactifying type IIB string theory on T^6 and then quotienting by the \mathbb{Z}_2 orientifold operation

$$\mathbb{Z}_2: \quad \Omega(-1)^{F_L} \mathcal{I}_6. \tag{2.1}$$

Here Ω is worldsheet orientation reversal, $(-1)^{F_L}$ is left-moving fermion parity,¹ and \mathcal{I}_6 is inversion of the T^6 . We take the coordinates of the T^6 to have unit periodicity, $x^m \cong x^m + 1$. The inversion \mathcal{I}_6 acts as $x^m \rightarrow -x^m$ on the T^6 , with fixed points at $x^m = 0, 1/2$ for $m = 1, \dots, 6$. These are the locations of 64 O3 planes.

For consistency of the model, a D3 charge cancellation condition on the T^6 must be satisfied. The condition is

$$2M + N_{\text{flux}} = 32, \tag{2.2}$$

where M is the number of independent D3 branes and N_{flux} is defined by

$$N_{\text{flux}} = \frac{1}{(2\pi)^4 \alpha'^2} \int_{T^6} H_{(3)} \wedge F_{(3)}. \tag{2.3}$$

This condition is the integral form of the $\tilde{F}_{(5)}$ Bianchi identity. We assume that there is no localized flux on the O3 planes. Then, on the \mathbb{Z}_2 covering space T^6 , there are M D3 branes plus M \mathcal{I}_6 image branes, and the flux is quantized as

$$H_{(3)}, F_{(3)} \in (2\pi)^2 \alpha' H^3(T^6, 2\mathbb{Z}). \tag{2.4}$$

¹The second factor is need to ensure a supersymmetric spectrum. For light states, i.e., those that oscillatorwise are massless, $(-1)^{F_L}$ acts as -1 on left-moving Ramond sector states and $+1$ on left-moving Neveu-Schwarz sector states.

The $2\mathbb{Z}$ quantization condition on the covering space T^6 is a \mathbb{Z} quantization condition on T^6/\mathcal{I}_6 . It guarantees integer periods of $F_{(3)}$ and $H_{(3)}$ over cycles in T^6/\mathcal{I}_6 that descend from half-cycles in T^6 .

Finally, the geometry is warped. The 10D string frame metric is

$$ds^2 = Z^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + Z^{1/2} ds_{T^6}^2, \tag{2.5}$$

where the warp factor satisfies the Poisson equation

$$-\nabla_{T^6}^2 Z = (2\pi)^4 \alpha'^2 g_s \left(\sum_I Q_I \frac{\delta^6(x - x_I)}{\sqrt{g_{T^6}}} + \frac{N_{\text{flux}}}{V_{T^6}} \right). \tag{2.6}$$

The dilaton profile is constant in the internal directions,

$$e^\phi = g_s. \tag{2.7}$$

In eq. (2.6), the sum runs over M D3 branes, M image D3 branes, and 64 O3 planes. The charge Q is normalized so that $Q_{\text{D3}} = 1$ and $Q_{\text{O3}} = -1/2$.

The simplest way to satisfy eq. (2.2) is with $M = 16$ D3 branes and no flux. Then the model preserves $\mathcal{N} = 4$ supersymmetry and is T-dual to type I on T^6 , via T-duality in the six torus directions.² More generally, we can trade off some or all of the D3 branes for quantized flux while still satisfying eq. (2.2). In the low energy effective field theory, the fluxes are the charges of a 4D $\mathcal{N} = 4$ gauged supergravity theory. In this theory, there is a superhiggs mechanism [5] that spontaneously breaks the supersymmetry to $\mathcal{N} < 4$. In terms of the torus length scale R , the superhiggs scale is α'/R^3 . (The factor of α' arises from the quantization condition (2.4), and the factor of R^3 from the volume of the 3-cycles carrying the flux.) For $R \gg 1/\sqrt{\alpha'}$, this scale is much lower than the Kaluza-Klein scale $1/R$, which in turn is much lower than the string scale.

2.2 $\mathcal{N} = 2$ flux and constraints on metric moduli

Up to $\text{SL}(2, \mathbb{Z})$ duality of the axion-dilaton and $\text{SL}(6, \mathbb{Z})$ change of lattice basis of the T^6 , the only known class of flux that preserves $\mathcal{N} = 2$ supersymmetry is [3, 17]

$$\begin{aligned} F_{(3)}/((2\pi)^2 \alpha') &= 2m(dx^4 \wedge dx^6 + dx^5 \wedge dx^7) \wedge dx^9, \\ H_{(3)}/((2\pi)^2 \alpha') &= 2n(dx^4 \wedge dx^6 + dx^5 \wedge dx^7) \wedge dx^8. \end{aligned} \tag{2.8}$$

Here m and n are positive integers satisfying the D3 charge cancellation condition

$$4mn + M = 16. \tag{2.9}$$

Note that interchange of m and n has the interpretation of S-duality followed by a 90° rotation in the 89-directions. We will return to this $m \leftrightarrow n$ duality in section 8.

The superhiggs mechanism for this class of flux was discussed recently in ref. [17]. The supersymmetry conditions require that $G_{(3)} = F_{(3)} - \tau_{\text{dil}} H_{(3)}$ be primitive and of Hodge

²For each pair of directions, the T-duality takes \mathcal{I}_{2p} to \mathcal{I}_{2p-2} and introduces another factor of $(-1)^{F_L}$ in eq. (2.1).

type (2, 1). (Primitivity means that $J \wedge G_{(3)} = 0$, where J is the Kähler form.) This constrains the torus to factorize as

$$T^6 = T^4_{\{4567\}} \times T^2_{\{89\}} \tag{2.10}$$

with respect to both Kähler and complex structure, and then imposes some additional constraints. A convenient parametrization of the metric moduli is obtained by writing the T^2 metric as

$$ds^2_{T^2} = \frac{v_3}{\text{Im } \tau_3} |dx^8 + \tau_3 dx^9|^2, \tag{2.11}$$

and then writing the T^4 as a flat fibration of $T^2_{\{45\}}$ over $T^2_{\{67\}}$:

$$ds^4_{T^4} = \frac{v_1}{\text{Im } \tau_1} |(dx^4 + a^4) + \tau_1(dx^5 + a^5)|^2 + \frac{v_2}{\text{Im } \tau_2} |dx^6 + \tau_2 dx^7|^2. \tag{2.12}$$

Here the flat connections a^4 and a^5 are constant 1-forms on $T^2_{\{67\}}$. The $\text{Im } \tau_i$ factors in the denominators are necessary so that the v_i are the volumes of the corresponding 2-tori. In terms of this parameterization, the further constraints on the metric moduli that follow from the supersymmetry conditions are

$$\tau_1 \tau_2 = -1, \quad (m/n) \tau_{\text{dil}} \tau_3 = -1, \quad (a^4)_7 = (a^5)_6, \tag{2.13}$$

with v_1, v_2, v_3 arbitrary.

2.3 The massless spectrum of T^6/\mathbb{Z}_2 with $\mathcal{N} = 2$ flux

The superhiggs mechanism involves more than just the metric moduli. Before taking into account the masses due to the flux, the massless bosonic fields preserved by the orientifold projection are

$$\text{Bulk: } \begin{cases} 1 \text{ 4D graviton } g_{\mu\nu}, \\ 12 \text{ 4D vectors } b_{(2)m\mu}, c_{(2)m\mu}, \\ 38 \text{ 4D scalars } c_{(4)mnpq}, g_{mn}, \tau_{\text{dil}} = c_{(0)} + ie^{-\phi}, \end{cases} \tag{2.14}$$

from the closed string sector, and

$$\text{Branes: } M \times \begin{cases} 1 \text{ 4D vector } A^I_\mu, \\ 6 \text{ 4D scalars } \Phi^{Im}, \end{cases} \tag{2.15}$$

from the open string sector. In 4D $\mathcal{N} = 4$ terms, these fields fill out the bosonic content of one gravity multiplet and $6 + M$ vector multiplets.

Now let us take into account the superhiggs mechanism due to the flux. The constraints (2.13) leave a total of 10 real metric moduli: v_1, v_2, v_3 , 2 independent τ , and 3 independent $(a^m)_n$. The kinetic terms for the $c_{(4)}$ scalars are proportional to

$$|dc_{(4)} - F_{(3)} \wedge b_{(2)} + H_{(3)} \wedge c_{(2)}|^2, \tag{2.16}$$

from the flux kinetic term $|\tilde{F}_{(5)}|^2$. (See ref. [2] for a detailed discussion.) The $c_{(4)}$ scalars are axionically coupled to the vectors $b_{(2)m\mu}$ and $c_{(2)n\mu}$ with charges given by $H_{(3)}$ and

$F_{(3)}$, respectively. One finds that 9 of the vectors eat 9 of the $c_{(4)}$ axions, leaving a total of 3 massless vectors and 6 massless $c_{(4)}$ axions [17]. In the open string sector, none of the massless D3 worldvolume fields is lifted by the flux. All together, these massless fields fill out the bosonic content of one $\mathcal{N} = 2$ gravity multiplet, $2 + M$ vector multiplets, and $3 + M$ hypermultiplets.

3. The O6/D6 dual of T^6/\mathbb{Z}_2

We will ultimately relate the above T^6/\mathbb{Z}_2 orientifold vacua to standard type IIA Calabi-Yau vacua by performing three T-dualities followed by a 9-10 circle swap (i.e., lifting to M theory on x^{10} and then compactifying on x^9).³ However, before attacking the full problem, let us first review the action of T-duality on NS flux in a simplified context [15].

3.1 Warm-up: the action of T-duality on NS flux

Consider a T^3 in the T^6/\mathbb{Z}_2 orientifold, such that $H_{(3)}$ through this T^3 is nonzero. For simplicity, we neglect the warp factor and take the metric on this T^3 to be

$$ds^2 = dx^2 + dy^2 + dz^2.$$

(It will be easy to include the warp factor and nontrivial metric when we discuss the case of interest in the next section.) Let the NS flux and potential be

$$H_{(3)} = N dx \wedge dy \wedge dz, \quad B_{(2)} = N x dy \wedge dz.$$

Now T-dualize in the z -direction. From the Buscher rules [18], the resulting metric and NS B -field are

$$ds^2 = dx^2 + dy^2 + (dz + N x dy)^2, \quad B_{(2)} = H_{(3)} = 0.$$

The T^3 metric has been replaced by an S^1 fibration, or equivalently U(1) principal bundle, over T^2 . The U(1) connection is $\mathcal{A} = N x dy$ and the curvature is $\mathcal{F} = d\mathcal{A} = N dx \wedge dy$. The topology of the fibration is characterized by the Chern class $[\mathcal{F}]$.

In other words, T-duality has interchanged the following two fibrations: (i) the explicit geometrical S^1 fibration of the isometry direction, and (ii) the formal \tilde{S}^1 fibration of connection $\tilde{\mathcal{A}} = -\int_{S^1} B_{(2)}$ and curvature $\tilde{\mathcal{F}} = \int_{S^1} H_{(3)}$. More generally, as long as there is an isometry that allows us to perform a T-duality, we can define both fibrations. Therefore, T-duality can always be interpreted as this sort of interchange. (See refs. [19, 17] for recent discussions.)

3.2 T^6/\mathbb{Z}_2 after three T-dualities

Now let us return to the problem of interest. Starting from the T^6/\mathbb{Z}_2 orientifold with $\mathcal{N} = 2$ flux, we can perform successive T-dualities in the x^4 , x^5 , and x^9 directions. The resulting 10D string frame metric is

$$ds^2 = Z'^{-1/2} (\eta_{\mu\nu} dx^\mu dx^\nu + ds_{T^3 \text{ fib}}^2) + Z'^{1/2} ds_{T^3 \text{ base}}^2, \tag{3.1a}$$

³The existence of this chain of dualities was first discussed in ref. [15] following suggestions by P. Berglund and N. Warner.

where

$$ds_{T^2 \text{ fib}}^2 = \frac{v_1'}{\text{Im } \tau_1'} |\eta^4 + \tau_1' \eta^5|^2 + R_9'^2 dx^{9^2}, \quad (3.1b)$$

$$ds_{T^2 \text{ base}}^2 = \frac{v_2}{\text{Im } \tau_2} |dx^6 + \tau_2 dx^7|^2 + R_8^2 dx^{8^2}. \quad (3.1c)$$

Here, $\eta^m = dx^m + \mathcal{A}^m$ for $m = 4, 5$, where the connections have curvatures

$$\mathcal{F}^4 = d\mathcal{A}^4 = 2ndx^6 \wedge dx^8 \quad \text{and} \quad \mathcal{F}^5 = d\mathcal{A}^5 = 2ndx^7 \wedge dx^8. \quad (3.2)$$

Just as a single T-duality gave us a circle fibration in the previous section, the three T-dualities give us an $(S^1)^3 = T^3$ fibration. However, since there were only two components of NS flux before the T-dualities (cf. eq. (2.8)), only two of the S^1 fibrations are nontrivial.

The primed metric moduli and string coupling after the T-dualities are related to unprimed T^6/\mathbb{Z}_2 quantities by

$$v_1' = (2\pi)^4 \alpha'^2 / v_1, \quad R_9' = (2\pi)^2 \alpha' / R_9, \quad \tau_1' = -1/\tau_1, \quad g_s' = g_s (2\pi)^6 \alpha'^3 / (R_4 R_5 R_9). \quad (3.3)$$

These moduli satisfy the new constraints

$$(n/m)v_1' = g_s' R_8, \quad \tau_1' = \tau_2. \quad (3.4)$$

The warp factor Z' is related to Z by averaging over the T-duality directions. It satisfies the equation

$$-\nabla_{T^3 \text{ base}}^2 Z' = 2\pi \sqrt{\alpha'} g_s' \left(\sum_I Q_I \frac{\delta^3(x - x_I)}{\sqrt{g_{T^3 \text{ base}}}} + \frac{N_{\text{flux}}}{V_{T^3 \text{ base}}} \right), \quad (3.5)$$

where now $Q_{D6} = 1$ and $Q_{O6} = -4$. Each T-duality adds one dimension to the O planes and D branes. After the three T-dualities, there are M D6 branes, M image D6 branes, and 8 O6 planes filling 4D spacetime and wrapping the T^3 fiber. The O6 planes are located at the the fixed points of \mathcal{I}_3 on the T_{base}^3 , where \mathcal{I}_3 takes (x^6, x^7, x^8) to $(-x^6, -x^7, -x^8)$. The solution to eq. (3.5) can be formally expressed as

$$Z' = 1 + 2\pi \sqrt{\alpha'} g_s' \sum_I Q_I G(x, x_I), \quad (3.6)$$

where the Green's function $G(x, x')$ for the Poisson equation on T_{base}^3 is defined by

$$-\nabla_{T^3}^2 G(x, x') = \frac{\delta^3(x - x')}{\sqrt{g_{T^3}}} - \frac{1}{V_{T^3}}, \quad \text{with} \quad \int_{T^3} d^3x \sqrt{g_{T^3}} G(x, x') = 0. \quad (3.7)$$

The leading constant of unity in eq. (3.6) has been chosen to ensure that the warp factor drops out of the metric in the limit $g_s' \rightarrow 0$.

In terms of Z' , the new dilaton profile is

$$e^{(\phi' - \phi'_0)} = Z'^{-3/4}, \quad \text{where} \quad e^{\phi'_0} = g_s'. \quad (3.8)$$

Finally, the only flux after the three T-dualities is

$$F_{(2)} = g_s'^{-1} *_3 dZ' - 2m(2\pi \sqrt{\alpha'}) (\eta^4 \wedge dx^7 - \eta^5 \wedge dx^6). \quad (3.9)$$

4. M theory lift and dual IIA Calabi-Yau compactification

4.1 Overview

In the strong coupling limit, the type IIA orientifold of the previous section lifts to a purely geometrical M theory compactification with no flux. Via the standard identifications, the $F_{(2)}$ flux determines the fibration of the x^{10} circle over the type IIA geometry, and the dilaton determines the size of this circle at each point. The only potentially singular objects in the lift are the O6 planes and D6 branes. However, as long as these objects are not coincident, each D6 brane lifts to a geometry that is locally a smooth Taub-NUT space times $\mathbb{R}^{6,1}$, and each O6 plane to a geometry that is locally a smooth Atiyah-Hitchin space times $\mathbb{R}^{6,1}$. Before the lift, the x^9 direction is not fibered, and does not appear in $F_{(2)}$. As we will see, the warp factors also cancel in the right way so that after the lift, neither the Minkowski metric nor the x^9 metric is mixed with the other directions through a multiplicative warp factor. Therefore, the M theory geometry factorizes as $\mathbb{R}^{3,1} \times S_{\{9\}}^1 \times X_6$, where X_6 is smooth. Since this background preserves $\mathcal{N} = 2$ supersymmetry, X_6 is necessarily a Calabi-Yau threefold. Compactifying on $S_{\{9\}}^1$ gives a standard Calabi-Yau compactification of type IIA string theory. This proves the desired result:

$$T^6/\mathbb{Z}_2 \text{ with } \mathcal{N} = 2 \text{ flux} \quad \leftrightarrow \quad \text{IIA on a Calabi-Yau } X_6.$$

The natural question that arises is Which Calabi-Yau threefold? We immediately know at least one piece of information. We computed the massless spectrum in the dual T^6/\mathbb{Z}_2 orientifold, and found that there were $M + 2$ vector multiplets and $M + 3$ hypermultiplets, where M was the number of D3 branes. This tells us that the Hodge numbers of X_6 are

$$h^{1,1} = h^{2,1} = M + 2. \tag{4.1}$$

From the D3 charge cancellation condition $M = 16 - 4mn$, the possible values of M are $M = 0, 4, 8, 12$, with multiplicities that are further distinguished by the choice of positive integers m and n .

Beyond the Hodge numbers, we would like to compute more detailed properties of the Calabi-Yau duals. What are the intersection numbers? Is there a fibration structure? Is X_6 simply-connected or is there a nontrivial fundamental group? A compact Calabi-Yau cannot have continuous isometries, but discrete isometries are allowed. Does X_6 have any discrete isometries? Finally, we identified the interchange of m and n as S-duality in the T^6/\mathbb{Z}_2 orientifold. What is the corresponding duality in the Calabi-Yau description?

4.2 Dual M theory and type IIA approximate metrics

In our conventions, the leading order relation between the 11D M theory metric and the type IIA dilaton, 1-form potential, and string frame metric is

$$ds_{11}^2 = e^{-2(\phi-\phi_0)/3} ds_{\text{IIA}}^2 + e^{4(\phi-\phi_0)/3} R_y^2 (dy + A)^2, \tag{4.2}$$

where $R_y = 2\pi\sqrt{\alpha'}g_s$ and $A = C_{(1)}/(2\pi\sqrt{\alpha'})$. Consequently, the leading order M theory metric from the lift of the O6/D6 dual of T^6/\mathbb{Z}_2 is

$$ds_{11}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + ds_6^2 + R_9^2 dx^{9^2}, \tag{4.3}$$

where

$$ds_6^2 = Z'^{-1} R_{10}^2 (dx^{10} + A)^2 + \frac{v_1'}{\text{Im } \tau_1'} |\eta^4 + \tau_1' \eta^5|^2 + Z' \left(\frac{v_2}{\text{Im } \tau_2} |dx^6 + \tau_2 dx^7|^2 + R_8^2 dx^{8,2} \right), \quad (4.4)$$

with coordinates identified modulo

$$(x^6, x^7, x^8, x^{10}) \rightarrow -(x^6, x^7, x^8, x^{10}). \quad (4.5)$$

The 1-forms η^4, η^5 and corresponding circle fibrations are defined after eq. (3.1). The fibration of the x^{10} circle is defined by

$$dA = R_{10}^{-1} *_3 dZ' - 2m(\eta^4 \wedge dx^7 - \eta^5 \wedge dx^6), \quad (4.6)$$

where $*_3$ is the Hodge star operator in the 3D base metric (3.1c) in the 678-directions. The 11D metric moduli satisfy the relations

$$v_1' = (m/n)R_8R_{10}, \quad \tau_1' = \tau_2. \quad (4.7)$$

From the M theory background, we can then compactify on $S_{\{9\}}^1$ to obtain the purely geometrical type IIA background

$$ds_{\text{IIA}}^2 = \eta_{\mu\nu} dx^m dx^n + ds_6^2, \quad (4.8)$$

with arbitrary string coupling. In this dual IIA compactification, the Regge slope and string coupling determine the M theory circle radius $R_9' = 2\pi\sqrt{\alpha'(\text{CY})}g_s(\text{CY})$. The relation to the 11D Planck scale is

$$M_{11}^{-1} = g_s^{1/3} \alpha'^{1/2} = g_s^{1/3}(\text{CY}) \alpha'^{1/2}(\text{CY}), \quad (4.9)$$

where $1/(2\kappa_{11}^2) = (2\pi)^{-8} M_{11}^9$ is the prefactor multiplying the 11D supergravity action.

4.3 Kähler form and (3,0) form

The metric (4.4) is *not* the Calabi-Yau metric on the smooth Calabi-Yau X_6 , but rather an approximation to it, in a sense that we will make precise in the next section. It is, however, a Calabi-Yau metric on the open space $Z' > 0$ for which the metric is positive definite. This is guaranteed by the fact that throughout the classical supergravity dualities we have satisfied the equations for an $\mathcal{N} = 2$ supersymmetric background. As an independent check of this claim, first note that there is a natural complex pairing of coordinates in which the $(1,0)$ -forms are

$$\begin{aligned} \eta^{z^1} &= \eta^4 + \tau_2 \eta^5, \\ dz^2 &= dx^6 + \tau_2 dx^7 \\ \eta^{z^3} &= (dx^{10} + A) - itZ' dx^8, \end{aligned} \quad (4.10)$$

where $t = R_8/R_{10}$. For future convenience, let us define the Kähler moduli h and s by

$$(v_1', v_2, R_8 R_{10}) = (\bar{m}h, 2s, \bar{n}h), \quad \text{where } (\bar{m}, \bar{n}) = (m, n)/\text{gcd}(m, n). \quad (4.11)$$

These parameters will turn out to be Kähler moduli relative to basis elements of integer cohomology. The Kähler form and (3,0) form for the complex structure (4.10) and metric ds_6^2 are

$$\begin{aligned}
 J &= \bar{m}h\eta^4 \wedge \eta^5 + 2sZ'dx^6 \wedge dx^7 + \bar{n}hdx^8 \wedge (dx^{10} + A), \\
 \Omega &= \left(\frac{\bar{m}h \cdot 2s \cdot \bar{n}h}{(\text{Im } \tau_2)^2 t} \right)^{1/2} \eta^{z^1} \wedge dz^2 \wedge \eta^{z^3}.
 \end{aligned}
 \tag{4.12}$$

Here, we have normalized Ω so that $\frac{i}{8}\Omega \wedge \Omega = \frac{1}{6}J \wedge J \wedge J = \text{Vol}_6$. Using eqs. (3.2) and (4.6), it is possible to show that $dJ = d\Omega = 0$. So, the intrinsic torsion [20] vanishes and the region $Z' > 0$ is Calabi-Yau.

4.4 Kähler and complex structure moduli

From the superhiggs mechanism in the original T^6/\mathbb{Z}_2 orientifold, it is possible to show that the only massless deformation of $C_{(1)}$ in the dual O6/D6 orientifold is $\delta C_{(1)} \propto dx^8$. This translates into a Calabi-Yau modulus $\delta A = adx^8$. Therefore, let us write

$$A = A_0 + adx^8, \tag{4.13}$$

and define

$$\tau^{-1} = a - it. \tag{4.14}$$

Then, eq. (4.10) can be alternatively written as

$$\begin{aligned}
 \eta^{z^1} &= dx^4 + \tau_2 dx^5 + \mathcal{A}^{z^1}, \\
 dz^2 &= dx^6 + \tau_2 dx^7, \\
 \eta^{z^3} &= dx^{10} + \tau^{-1} dx^8 + \mathcal{A}^{z^3},
 \end{aligned}
 \tag{4.15}$$

where

$$\begin{aligned}
 \mathcal{F}^{z^1} &= d\mathcal{A}^{z^1} = 2ndz^2 \wedge dx^8, \\
 \mathcal{F}^{z^3} &= d\mathcal{A}^{z^3} = \partial_8 \hat{Z} dx^6 \wedge dx^7 - 2m(\eta^4 \wedge dx^7 - \eta^5 \wedge dx^6),
 \end{aligned}
 \tag{4.16}$$

and \hat{Z} is defined by

$$Z' = 1 + \left(\frac{\bar{n}h}{2s} \right) \hat{Z}. \tag{4.17}$$

The quantity \hat{Z} satisfies the rescaled Poisson equation

$$\left(\partial_8^2 + \frac{\bar{n}ht}{2s} \nabla_{\hat{T}^2}^2 \right) \hat{Z} = \sum_I Q_I (\delta^3(x - x_I) - 1), \quad \int \hat{Z} d^3x = 0, \tag{4.18}$$

where \hat{T}^2 is the torus obtained from $T_{\{6,7\}}^2$ by rescaling to unit area. In the limit of small relative Kähler modulus $h/s \ll 1$, the warp factor Z' is positive and the metric (4.4) is positive definite everywhere except in a small neighborhood of the \mathcal{I}_3 fixed loci.

The complete list of Kähler and complex structure moduli is

$$\begin{array}{ll}
 \text{Complex structure Moduli} & \tau, \tau_2, \text{ and } M \text{ complex dof in } \hat{Z} \quad (2 + M \text{ total}) \\
 \text{Kähler moduli:} & h, s, \text{ and } M \text{ real dof in } \hat{Z} \quad (2 + M \text{ total}),
 \end{array}$$

in agreement with the earlier result for the Hodge numbers that we deduced from the number of massless vector and hyper multiplets in the T^6/\mathbb{Z}_2 orientifold.

5. Approximate versus exact M theory lifts of orientifolds

The standard relation (4.2) between the 11D metric and type IIA supergravity background assumes that the 11d metric has an isometry in the direction used in the dimensional reduction. If this is not the case, then the relation does not give the full 11D metric, but rather its truncation to the lowest Fourier mode around the M theory circle. To gain insight on the relation between the exact Calabi-Yau duals of T^6/\mathbb{Z}_2 and the leading order description given in the previous section, it is helpful to first consider the simpler case of an O6 plane and/or D6 branes in flat space. The description below draws heavily on ref. [16].

5.1 M theory lift of a single D6 brane or O6 plane

For the lift of a single D6 brane or O6 plane in flat space, the M theory background resulting from eq. (4.2) is the purely geometrical background given by $\mathbb{R}^{6,1}$ times a 4D space with metric

$$ds_4^2 = Z^{-1} R_{10}^2 (dx^{10} + A)^2 + Z ds_{\mathbb{R}^3}^2, \quad (5.1a)$$

where

$$dA = R_{10}^{-1} *_3 dZ, \quad \text{and} \quad Z = 1 + \frac{R_{10}}{8\pi^2} \frac{Q}{|\vec{x}|} \quad \text{with} \quad \vec{x} \in \mathbb{R}^3. \quad (5.1b)$$

For a D6 brane, we have $Q = 1$ and this defines a smooth Taub-NUT space. Nothing is lost in the truncation to lowest Fourier mode since the Taub-NUT space has a $U(1)$ isometry around the x^{10} fiber. The core of the D6 brane lifts to the point at which this fiber shrinks to zero radius.

For an O6 plane, $Q = -4$. Eq. (5.1) together with the coordinate identification $(\vec{x}, x^{10}) \cong -(\vec{x}, x^{10})$ defines the large radius approximation to an Atiyah-Hitchin space [21]. It is singular at small \vec{x} . However, this just reflects the fact that we have discarded all higher Fourier modes in the x^{10} direction. The complete Atiyah-Hitchin geometry is smooth.

The approximation becomes progressively worse as $|\vec{x}|$ is decreased, until at small but finite \vec{x} the inverse warp factor diverges and the M theory circle decompactifies. In type IIA language, D0 branes become light near the O6 planes, where $e^\phi \rightarrow \infty$. In this region, the truncation of the low energy effective field theory to the 10D type IIA supergravity multiplet is a poor approximation. It is necessary to include the RR charged fields from the complete tower of D0 bound states to reliably describe the local physics near the O6 planes.

The metric (5.1) also has an interpretation as the 1-loop moduli space metric of a D2 brane probe [22]. In the O6 case, this metric is corrected by 3D instantons to the smooth Atiyah-Hitchin metric. However, this interpretation does not persist as an exact quantitative correspondence in the case of primary interest in this paper with only half as much supersymmetry [23]. Therefore, we will not pursue it here.

To further set the stage for the interpretation of the approximate Calabi-Yau metrics of section 4, we now review the A_{M-1} and approximate D_M metrics resulting from leading order M theory lift of M D6 branes (section 5.2) and M D6 branes in the presence of an O6 plane (section 5.3), respectively. In either case, the metric is again given by eqs. (5.1a), (5.1b), however the expressions for the warp factor differ.

5.2 M theory lift of M D6 branes

For the lift of M D6 branes at positions \vec{x}_I in the transverse \mathbb{R}^3 , the warp factor is

$$Z = 1 + \sum_{I=1}^M Z^I, \quad \text{where} \quad Z^I = \frac{R_{10}}{8\pi^2} \frac{1}{|\vec{x} - \vec{x}_I|}. \quad (5.2)$$

eq. (5.1a), (5.1b) is then the metric for a multicentered Taub-NUT space [24]. The S^1 fiber shrinks over each center \vec{x}_I on the \mathbb{R}^3 base. In this geometry, we obtain 2-cycles with the topology of spheres from the fibration over curves in \mathbb{R}^3 connecting any pair of centers x_I, x_J . In the notation of ref. [16], if we let $S_{I,J}$ denote the homology class of a sphere formed from this pair of centers, then the set $\{S_{1,2}, S_{2,3}, \dots, S_{M-1,M}\}$ is a basis for H_2 . The basis forms a “chain of sausage links,” in which neighboring spheres intersect in 1 point: $S_{I-1,I} \cdot S_{I,I+1} = 1$. Also, $S_{I,I+1}^2 = -2$ since any two representatives of $S_{I,I+1}$ intersect in the points x_I, x_{I+1} . The minus sign is due to orientation.⁴ The intersection matrix is minus the Cartan matrix of $SU(M)$, and the space describes the resolution of an A_{M-1} singularity.

There is also a Poincaré dual description. There exists one L_2 harmonic form for each center \vec{x}^I [27]:

$$F^I = \left(Z^I/Z \right)_{,m} (-dx^m \wedge (dx^{10} + A) + \frac{1}{2} Z R_{10}^{-1} \text{Vol}_{\mathbb{R}^3}{}^m{}_{np} dx^n \wedge dx^p) \\ \stackrel{\text{locally}}{=} d\left(A^I - (Z^I/Z)(dx^{10} + A) \right). \quad (5.3a)$$

Here A^I is defined by eq. (5.1b) with Z replaced by Z^I . The F^I are anti-selfdual and have intersections

$$\int F^I \wedge F^J = -\delta^{IJ}. \quad (5.4)$$

The harmonic forms $\omega^{I,J} = F^I - F^J$ have A_{M-1} intersection matrix and are representatives of the cohomology classes dual to the spheres $S_{I,J}$. In discussing the Calabi-Yau duals of T^6/\mathbb{Z}_2 , it will initially be more convenient to work with an analogous cohomology description than with homology.

5.3 M theory lift of M D6 branes near an O6 plane

For the lift of M D6 branes plus one O6 plane at the origin, the warp factor is

$$Z = 1 - \frac{R_{10}}{8\pi^2} \frac{4}{|\vec{x}|} + \sum_{I=1}^M (Z^J + Z^{I'}), \quad (5.5a)$$

where

$$Z^I = \frac{R_{10}}{8\pi^2} \frac{1}{|\vec{x} - \vec{x}_I|} \quad \text{and} \quad Z^{I'} = \frac{R_{10}}{8\pi^2} \frac{1}{|\vec{x} + \vec{x}_I|}, \quad (5.5b)$$

⁴An alternative description of this minus sign is as follows [25, 26]. A multicentered Taub-NUT space is a Calabi-Yau 2-fold, so its first Chern class vanishes. One can then show from the adjunction formula that the self-intersection number of a genus g holomorphic curve is $2g - 2$. The $g = 0$ rational curves are isolated, but formally have self-intersection -2 for agreement with the $g > 0$ formula.

and there is a \mathbb{Z}_2 coordinate identification $(\vec{x}, x^{10}) \cong -(\vec{x}, x^{10})$. In this case, for each pair of centers \vec{x}_I, \vec{x}_J , we obtain a sphere $S_{I,J}$ from the S^1 fibration over a curve in \mathbb{R}^3 connecting the two centers, together with its \mathbb{Z}_2 image. In addition, for each center \vec{x}_I and image center $\vec{x}_{J'} = -\vec{x}_J$, we obtain a sphere $S_{I,J'}$ from the S^1 fibration over a curve in \mathbb{R}^3 connecting \vec{x}_I and $\vec{x}_{J'}$ together with its \mathbb{Z}_2 image. A basis of H_2 is obtained from the classes $S_{I,I+1}$ for $I = 1, \dots, M-1$, together with $S_{M-1,M'}$. The intersection numbers in this basis can be computed as in the previous case. The intersection matrix is minus the Cartan matrix of $SO(2M)$, and the metric is a large radius approximation to that of a resolved D_M singularity.

The metric is singular at sufficiently small $|\vec{x}|$. However, we can choose representative cycles that avoid the bad regions where $Z \leq 0$. Therefore, we obtain correct topological data (intersection numbers) from the singular leading order lift. The full lift smoothly excises the $Z \leq 0$ regions just as an Atiyah-Hitchin space did for the lift of an O6 plane alone.

There is again a Poincaré dual description. In this case, the harmonic representative of the cohomology class dual to $S_{I,J}$ is $\omega_{I,J} = (F^I - F^J) - (F^{I'} - F^{J'})$, and that of the cohomology class dual to $S_{I,J'}$ is $\omega_{I,J'} = (F^I - F^{J'}) - (F^{I'} - F^J)$. Here, F^I is given by eq. (5.3) and $F^{I'}$ by the same formula with I replaced by I' . Explicit calculation shows that these cohomology classes give exactly a D_M intersection matrix. It is somewhat counterintuitive that we obtain the correct result, since the domain of integration includes the unreliable $Z \leq 0$ regions. However, this result is expected since intersection numbers must be the same for other cohomology representatives that are supported away from the $Z \leq 0$ regions.

6. (Co)homology, intersections, and fibration structure of the CY₃ duals

In section 4, we derived an approximate description of Calabi-Yau manifolds X_6 dual to T^6/\mathbb{Z}_2 with $\mathcal{N} = 2$ flux. The approximate metric (4.4) contains a function Z' , which determines the overall scaling of the metric in the x^{10} and the $x^{6,7,8}$ directions. We showed that $Z' = 1 + (\bar{n}h/2s)\hat{Z}$, with $\hat{Z} = \mathcal{O}(1)$ as $h/s \rightarrow 0$, so that by tuning h/s , we can make the unreliable $Z' \leq 0$ regions smaller and smaller. In the limit $h/s \rightarrow 0$, the leading order metric becomes an arbitrarily good approximation at most points. Moreover, nothing special happens at the bad loci (fixed points of \mathcal{I}_3); these loci locally have the geometry of an Atiyah-Hitchin space times \mathbb{R}^2 in the exact description. Therefore, we expect the homology and cohomology to be reliably computable in the leading order description of the duality, just as in the previous section.

6.1 Cohomology and intersection numbers of the Calabi-Yau duals

Expressing the Kähler form (4.12) in terms of \hat{Z} , we have

$$J = s\omega_S + h\omega_H, \tag{6.1}$$

where

$$now_H = \bar{m}\eta^4 \wedge \eta^5 + \bar{n}\hat{Z}dx^6 \wedge dx^7 + \bar{n}dx^8 \wedge (dx^{10} + A), \quad (6.2a)$$

$$\omega_S = 2dx^6 \wedge dx^7. \quad (6.2b)$$

As in section 5, we obtain one harmonic form for each D brane before the duality:

$$\omega_I = F^I - F^{I'}, \quad (6.2c)$$

where now

$$F^I = (Z^I/Z'),_m \left(-dx^m \wedge (dx^{10} + A) + \frac{1}{2} Z' R_{10}^{-1} \text{Vol}_{\mathbb{R}^2}{}^m{}_{np} dx^n \wedge dx^p \right) + 2m(Z^I/Z')(\eta^4 \wedge dx^7 - \eta^5 \wedge dx^6). \quad (6.3)$$

Here, $Z^I = G(x, x_I)$ in the notation of section 3.2. If we define A^I via $dA^I = R_{10}^{-1} *_3 dZ^I$, then the F_I again have the local expression (5.3a). In the leading order description (4.4), $\omega_H, \omega_S, \omega_I$ form a basis for (the free part of) $H^2(X_6, \mathbb{Z})$. The corresponding intersection numbers are $A \cdot B \cdot C = \frac{1}{2} \int \omega_A \wedge \omega_B \wedge \omega_C$, where the factor of 1/2 is due to the \mathbb{Z}_2 identification of coordinates (4.5). Letting \mathcal{E}_I denote the Poincaré dual of ω_I , one finds that

$$H^2 \cdot S = 2\bar{m}\bar{n}, \quad H \cdot \mathcal{E}_I \cdot \mathcal{E}_J = -\bar{m}\delta_{IJ}, \quad \text{others} = 0. \quad (6.4)$$

To check that the harmonic forms $\omega_H, \omega_S, \omega_I$ are correctly normalized representatives of integer cohomology, we use the fact that for a IIA Calabi-Yau compactification with no discrete torsion, the continuous NS B -field moduli take values in $(2\pi)^2 \alpha' H^2(X_6, \mathbb{R}) / H_{\text{free}}^2(X_6, \mathbb{Z})$. Since the periodicities of the B -field moduli can be deduced from the periodicities of the dual moduli in the T^6/\mathbb{Z}_2 orientifold, the lattice $H^2(X_6, \mathbb{Z})$ follows.

To verify the normalization of ω_S , we note that $(c_{(4)4567})_{T^6/\mathbb{Z}_2} = (2\pi)^2 \alpha' (b_{(2)67})_{\text{CY}}$. In this case, the dual periodicity is $c_{(4)4567} \cong c_{(4)4567} + 2 \cdot (2\pi)^4 \alpha'^2$. The factor of 2 is related to the orientifold projection.⁵ Therefore, $2dx^6 \wedge dx^7$ generates a maximal 1D sublattice of $H_{\text{free}}^2(X_6, \mathbb{Z})$ in the leading order description. In the next section, we will interpret the Calabi-Yau manifold X_6 as a fibration over $\mathbb{P}^1 \cong T_{\{67\}}^2/\mathcal{I}_2$. The form ω_S can then be interpreted as the pullback of the generator of $H^2(T^2/\mathcal{I}_2, \mathbb{Z})$ to X_6 .

For ω_H , the check is slightly more complicated, since we need to take into account moduli constraints. In this case, $(b_{(2)45})_{\text{CY}} = (2\pi)^2 \alpha' (c_{(0)})_{T^6/\mathbb{Z}_2}$ and $(b_{(2)810})_{\text{CY}} = (2\pi)^2 \alpha' (\text{Re}(-1/\tau_3))_{T^6/\mathbb{Z}_2}$. The moduli $c_{(0)}$ and $\text{Re}(-1/\tau_3)$ have unit periodicity in the absence of flux; however, the flux imposes the moduli constraint

$$(n/m)\tau_{\text{dil}} = -1/\tau_3 \quad \Leftrightarrow \quad nb_{(2)45} = mb_{(2)810}. \quad (6.5)$$

⁵In ref. [2], this factor of 2 follows from formulae in sections III and IV relating type T^6/\mathbb{Z}_2 in the absence of flux to type I, and subsequently to the heterotic string on T^6 . Another way to derive it is to require that $\exp(i \oint c_{(4)})$ be single-valued for D3 instantons wrapping half-cycles on T^6 that descend to cycles on T^6/\mathcal{I}_6 in the T^6/\mathbb{Z}_2 orientifold.

This moduli constraint implies the combined periodicity

$$(b_{(2)45}, b_{(2)810}) \cong (b_{(2)45}, b_{(2)810}) + (2\pi)^2 \alpha' (\bar{m}, \bar{n}), \tag{6.6}$$

where, as above, we define $(\bar{m}, \bar{n}) = (m, n) / \text{gcd}(m, n)$. Comparing eq. (6.6) to (6.2b), we see that ω_H is correctly normalized to generate a maximal 1D sublattice of $H_{\text{free}}^2(X_6, \mathbb{Z})$ in the leading order description.

For the ω_I , the check is cumbersome but straightforward. We omit the details here. The harmonic forms (6.2c) are correctly normalized as a consequence of the periodicities of the D3 brane worldvolume scalars in the T^6/\mathbb{Z}_2 orientifold, or equivalently, the periodicities of the D6 brane worldvolume Wilson lines in the O6/D6 dual.

The reader might wonder why there are only two explicit Kähler moduli in eq. (6.1), but $2+M$ harmonic forms. The reason is that Kähler deformations in the ω_I directions can always be absorbed into the warp factor. Taking $J \rightarrow J + t^I \omega_I$ for small t^I deforms J to a nearby cohomology class whose harmonic representative in the corresponding deformed metric takes exactly the same form as (6.1), but with slightly displaced sources entering into the Poisson equation for Z' .

6.2 Fibration structure

The qualitative form of X_6 that arose in section 4 from the duality chain is as follows. We first fiber $T_{\{4,5\}}^2$ over $T_{\{678\}}^3$, then fiber $S_{\{10\}}^1$ over the resulting geometry, and finally quotient by the \mathbb{Z}_2 coordinate identification (4.5). This means that in the leading order description, the geometry is a fibration over $T_{\{678\}}^3/\mathcal{I}_3$. Since the latter is nonorientable, this fibration is not very useful. However, we can instead view the geometry as a fibration over $T_{\{67\}}^2/\mathcal{I}_2 \cong \mathbb{P}^1$, where \mathcal{I}_2 takes (x^6, x^7) to $-(x^6, x^7)$.⁶ The field strengths of all connection 1-forms ($\mathcal{A}^4, \mathcal{A}^5, A$, or equivalently, $\mathcal{A}^{z^1}, \mathcal{A}^{z^3}$) restrict trivially to the subspace $(x^6, x^7) = \text{constant}$. Therefore, the generic fiber is the product $S_{\{10\}}^1 \times S_{\{8\}}^1 \times T_{\{45\}}^2 = T^4$, with no twists. We can trust this result since our approximate description can be made valid to arbitrary accuracy away from the \mathcal{I}_3 fixed loci, and therefore suffices for describing the generic fiber. In fact, this T^4 has an additional structure that makes it an abelian surface.

The Kähler form on the fiber is

$$J_{\text{fib}} = h\omega, \tag{6.7}$$

where h is the same Kähler modulus as above and ω is the 2-form

$$\omega = \bar{m} dy^4 \wedge dy^5 + \bar{n} dy^8 \wedge dy^{10} \tag{6.8}$$

on the fiber. Here, y^m are coordinates with unit periodicity $y^m \cong y^m + 1$, which may or may not be the same as the coordinates x^m restricted to a given point in the base, depending upon the choice of gauge for the connections. The possible values of (\bar{m}, \bar{n}) are $(1, 1), (1, 2), (1, 3), (1, 4)$, together with $\bar{m} \leftrightarrow \bar{n}$ interchanges.

⁶The space T^3/\mathcal{I}_3 is an S^1 fibration over T^2/\mathcal{I}_2 with singular fibers at the four \mathcal{I}_2 fixed points.

In general, when T^{2d} has a Kähler form proportional to

$$\omega = \sum_{i=1}^d a_i dy^{2i-1} \wedge dy^{2i}, \quad \text{with } a_i \mid a_{i+1}, \quad (6.9)$$

we can apply the Kodaira Embedding Theorem, which states that [28]: *a compact complex manifold X is an algebraic variety — i.e., is embeddable in a projective space — if and only if it has a closed $(1,1)$ -form ω whose cohomology class $[\omega]$ is rational.*⁷

In the case that $X = T^{2d}$, the variety is called an abelian variety and the cohomology class $[\omega]$ is called a polarization. The integers a_i are the invariants of the polarization [28]. In our case, $d = 2$, so the torus is an abelian surface, and the Calabi-Yau an abelian surface fibration over \mathbb{P}^1 .

6.3 Homology

We are now in a position to identify the homology classes Poincaré dual to eq. (6.2a) (6.2b), (6.2c). The easiest to identify is the class S dual to ω_S , which is the class of the abelian surface fiber.

Since the fiber is an algebraic variety, it has at least a hyperplane class of divisors (obtained by intersecting the variety with codimension 1 hyperplanes of the projective space). The class H dual to ω_H is the fibration of the hyperplane class of T^4 over the \mathbb{P}^1 base.

The remaining homology classes are very similar to those of section 5.3. The leading order description on the \mathbb{Z}_2 cover is as follows. The class $\mathcal{E}_I - \mathcal{E}_J$ dual to $\omega_I - \omega_J$ is represented by the fibration of $T_{\{45\}}^2 \times S_{\{10\}}^1$ over a real curve in $T_{\{678\}}^3$ connecting x_I to x_J , plus its \mathbb{Z}_2 image fibered over the curve connecting $x_{I'}$ to $x_{J'}$. This 4-cycle is the analog of the 2-sphere $S_{I,J}$ of section 5.3: the circle $S_{\{10\}}^1$ shrinks at the endpoints, but now $T_{\{45\}}^2$ (which does not shrink) is fibered as well. Similarly, the class $\mathcal{E}_I + \mathcal{E}_J$ dual to $\omega_I + \omega_J$ is represented by the $T_{\{45\}}^2 \times S_{\{10\}}^1$ fibration over a real curve in $T_{\{678\}}^3$ connecting x_I to $x_{J'}$, plus its \mathbb{Z}_2 image. This is the analog of $S_{I,J'}$ of section 5.3. A complete basis of $H_2(X_6, \mathbb{Z})$ in the leading order description is obtained from $\mathcal{E}_I - \mathcal{E}_{I+1}$ for $I = 1, \dots, M - 1$ and $\mathcal{E}_{M-1} + \mathcal{E}_M$. On a divisor of class H , the intersection matrix of the curves corresponding to the divisors $\mathcal{E}_I - \mathcal{E}_{I+1}$ ($I = 1, \dots, M - 1$) and $\mathcal{E}_{M-1} + \mathcal{E}_M$ is proportional to the D_M Cartan matrix (cf. section 5.3 and eq. (6.4)). This reflects the fact that there is enhanced $SO(2M)$ gauge symmetry when all M D3 branes (and M image D3 branes) coincide with an O3 plane in the dual T^6/\mathbb{Z}_2 orientifold.

Let us now give an alternative description of the last paragraph in terms of the abelian surface fibration over \mathbb{P}^1 and the degenerations of this fibration. At each point $(x^6, x^7) = (x_I^6, x_I^7)$ on the $\mathbb{P}^1 = T_{\{67\}}^2/\mathcal{I}_2$ base, the T^4 degenerates to $T_{\{45\}}^2 \times I_1$. The I_1 factor is $T_{\{8,10\}}^2|_{(x_I^6, x_I^7)}$, which has a single node at $x^8 = x_I^8$ where the $S_{\{10\}}^1$ circle degenerates to a point (since $Z'^{-1} \rightarrow 0$ in eq. (4.3)). Note that the same cycle degenerates in each I_1 . The class $\mathcal{E}_I - \mathcal{E}_J$ is represented by the fibration of $T_{\{45\}}^2$ times the degenerating cycle of $T_{\{8,10\}}^2$

⁷An algebraic variety is locally the common zero locus of set of homogeneous polynomials in a complex projective space.

over a real curve in the \mathbb{P}^1 base. In the full Calabi-Yau geometry, there are expected to be additional degenerations from the Atiyah-Hitchin-like regions that excise the $Z' \leq 0$ regions of the leading order description. The class $\mathcal{E}_I + \mathcal{E}_J$ in the full geometry is represented by a $T_{\{45\}}^2 \times S^1$ fibration over a real 1D locus in the \mathbb{P}^1 that terminates at x_I and x_J as well as at the locations of other fiber degenerations in the Atiyah-Hitchin-like regions. In the leading order description, this locus is a real curve connecting x_I to one of the four \mathcal{I}_2 fixed points, plus a curve connecting x_J to the same fixed point; the S^1 that degenerates is $S_{\{10\}}^1$. In the full Calabi-Yau geometry, the description is the same in the reliable regions; however, in the Atiyah-Hitchin-like regions, the 1D locus in the \mathbb{P}^1 base can become a more complicated junction of curves over which different S^1 cycles of $T_{\{8,10\}}^2$ degenerate.

We can gain some intuition for the geometry in the Atiyah-Hitchin-like regions as follows.⁸ Consider an elliptic realization of the D_M space X_4 of section 5.3. In the F theory limit in which the area of the fiber is scaled to zero, M theory on X_4 becomes a type IIB orientifold with one O7 plane and M D7 branes, i.e., the IIB background that is T-dual to the O6/D6 background of section 5.3. In the F theory description, each D7 brane corresponds to an I_1 degeneration of the elliptic fibration in which a $(1, 0)$ cycle of the fiber degenerates. The O7 plane resolves to a pair of (p, q) seven-branes with $(p, q) = (1, 1)$ and $(1, -1)$.⁹ Each (p, q) seven-brane corresponds to an I_1 degeneration of the elliptic fibration of X_4 in which a (p, q) cycle of the fiber degenerates. The rational curves $S_{I,J}$ come from the fibration of the $(1, 0)$ -cycle of the fiber over a real curve in the base connecting two of the locations of $(1, 0)$ I_1 degenerations. The rational curves $S_{I,J'}$ come from the S^1 fibration over an H-shaped junction in the base, where x_I and x_J are the endpoints on the right edge of the letter H, and the locations of the two (p, q) degenerations are the endpoints on the left edge of the letter H. In the IIB description, this H-shaped junction is a string junction with two $(1, 0)$ strings on the right, a $(2, 0)$ string in the middle, and $(1, 1), (1, -1)$ strings on the left. (The fundamental strings terminate on D7 branes, and (p, q) strings on (p, q) seven-branes.) In the M theory description, the junction fattens out into a rational curve $S_{I,J'}$, where the S^1 that is fibered over a (p, q) segment of the junction in the base is the (p, q) -cycle of the elliptic fibration.

We expect something similar to happen in the Calabi-Yau geometry, with the 2-cycles of the previous paragraph replaced by 4-cycles due to the extra two dimensions from $T_{\{45\}}^2$. It is tempting to conjecture that in each of the four Atiyah-Hitchin-like regions of the Calabi-Yau, the T^4 fiber degenerates to $T_{\{45\}}^2 \times I_1$ at two points on the base \mathbb{P}^1 . (Each pair of points would coalesce in the leading order description to one of the four fixed point of \mathcal{I}_2 in $\mathbb{P}^1 = T_{\{67\}}^2/\mathcal{I}_2$.) Intuitively, the (m, n) -dependent global topology of the abelian

⁸The discussion in this paragraph is based on ref. [29]. We refer the reader to the D_M section of ref. [29] for a more complete exposition of the ideas presented here.

⁹This can be seen in the Seiberg-Witten theory [30] on a D3 brane probing an O7 plane [31]. Classically, there is a point of enhanced SU(2) symmetry where the D3 brane is coincident with the O7 plane. Nonperturbatively, there is no enhanced SU(2) point. The point resolves into a massless $(1, 1)$ dyon point and a separate massless $(1, -1)$ dyon point, corresponding to a D3 brane coincident with either of the two types of (p, q) seven-branes. (The massless hypermultiplets come from strings stretched between the D3 brane and seven-brane.)

surface fibration should not effect the local geometry in these regions. However, we do not yet have a sufficiently complete understanding of the geometry in the Atiyah-Hitchin-like regions to motivate this conjecture further. We leave this task for the future.

6.4 Check of fibration result

As a check that the Calabi-Yau duals of T^6/\mathbb{Z}_2 with $\mathcal{N} = 2$ flux are abelian surface fibrations over \mathbb{P}^1 , recall the following theorem due to Oguiso [32], which is reviewed in refs. [33, 25]: *Let X be a minimal Calabi-Yau threefold. Let D be a nef divisor on X . If the numerical D -dimension of D equals one then there is a fibration $\Phi : X \rightarrow W$, where W is \mathbb{P}^1 and the generic fiber is either a K3 surface or an abelian surface.*

The numerical D -dimension of a divisor is simply the largest integer n such that $D^n \neq 0$. From the intersection numbers (6.4), we see that $S^2 = 0$, so S is a divisor of numerical D -dimension 1. A divisor is nef (numerically effective) if $D \cdot C \geq 0$ for any algebraic curve C . The space of nef divisors arises as the closure of the space of ample divisors ($D \cdot C > 0$), which is important in realizing projective embeddings. However, nefness is also a weaker analog of effectiveness. A divisor is effective if it is a formal linear combination of irreducible analytic hypersurfaces with nonnegative coefficients [28].

We now give a heuristic argument for why S should be effective. Due to the pairing (4.15), it is natural to attempt to define complex coordinates via

$$z^1 = x^4 + \tau_2 x^5, \quad z^2 = x^6 + \tau_2 x^7, \quad z^3 = x^8 + \tau x^{10}.$$

However, as a consequence eq. (4.16), of these are not holomorphic coordinates. For example, in the gauge $\mathcal{A}^{z^1} = 2nz^2 dx^8$, the globally defined $(1,0)$ -form η^{z^1} is

$$dz^1 + 2nz^2 dx^8 = d(z^1 - 2nx^8) + 2n(z^2 + 1)dx^8 = d(z^1 - 2n\tau_2 x^8) + 2n(z^2 + \tau_2)dx^8,$$

from which we deduce the nonholomorphic identifications (i.e., transition functions)

$$(z^1, z^2, z^3) \cong (z^1 - 2n \operatorname{Re}(z^3), z^2 + 1, z^3) \cong (z^1 - 2n\tau_2 \operatorname{Re}(z^3), z^2 + \tau_2, z^3).$$

Eq. (4.16) implies that our tentative definitions of z^1 and z^3 need to be modified in order for the identifications to be holomorphic. However, it does not imply that the definition of z^2 needs to be modified. Assuming that it is not modified, a divisor of class S is given by the holomorphic equation $z^2 = \text{constant}$, and is therefore an effective divisor. It then follows from Oguiso’s theorem that X_6 a fibration over \mathbb{P}^1 .

To decide whether the generic fiber is K3 or an abelian surface, we use the following expression [33, 25] for the Euler number of S , which follows from $S^2 = 0$ together with the fact that both K3 and T^4 have vanishing first Chern class:

$$\chi(S) = \int_S c_2(S) = S \cdot c_2(X_6). \tag{6.10}$$

If $\chi(S) = 0$ then the fiber is an abelian surface, and if $\chi(S) = 24$ it is a K3 surface.

It is easy to evaluate the right hand side, since the quantity $S \cdot c_2(X_6)$ is a familiar object that appears in the genus 1 topological A-model amplitude [34] on X_6 ,

$$F_1 \propto \sum_{\alpha=1}^{h^{1,1}(X_6)} (D_\alpha \cdot c_2(X_6)) t^\alpha + \text{worldsheet instantons.} \tag{6.11}$$

Here, the sum runs over a basis of $H^{1,1}$, with corresponding Kähler moduli t^α and Poincaré dual divisors D_α . This topological string amplitude enters into a curvature squared term proportional to $\text{Re} \int F_1 \text{tr}(R - R^*)^2$ in the 4D low energy effective action. In the T^6/\mathbb{Z}_2 orientifold, as in type I or the heterotic string on T^6 , F_1 is determined by Green-Schwarz anomaly cancellation¹⁰ to be

$$F_1 \propto \tau_{\text{dil}} + \text{quantum corrections,} \tag{6.12}$$

where $\tau_{\text{dil}} = c_{(0)} + i/g_s$. We will see in the next section that $1/g_s$ of the T^6/\mathbb{Z}_2 orientifold maps to the Kähler modulus h of the Calabi-Yau X_6 , i.e., there is no s -dependence in the first term of eq. (6.11). Therefore, $\chi(S) = S \cdot c_2(X_6) = 0$, and the fiber is an abelian surface.

7. Torsion 1-cycles and discrete isometries of X_6

We saw in eq. (2.16) that the $c_{(4)}$ scalars of the T^6/\mathbb{Z}_2 orientifold are axionically coupled to the U(1) gauge bosons $b_{(2)m\mu}$ and $c_{(2)m\mu}$, with charges given by the 3-form flux. When the coupling is nonminimal and a component of $c_{(4)}$ couples to $Nb_{(2)m\mu}$ or $Nc_{(2)m\mu}$, the corresponding U(1) gauge symmetry is only partially broken, leaving a residual discrete gauge symmetry \mathbb{Z}_N . These discrete gauge symmetries contain information about the torsion cycles and discrete isometries of the Calabi-Yau duals, X_6 .

Let us focus on the torsion 1-cycles and discrete isometries. Suppose that there is a \mathbb{Z}_N 1-cycle γ , i.e., a cycle such that $N\gamma$ is trivial in $H_1(X_6, \mathbb{Z})$. This means that a fundamental string can be wrapped on γ with winding number conserved modulo N . The \mathbb{Z}_N gauge field that couples to the winding charge is $\int_\gamma dx^m b_{(2)m\mu}$. Similarly, suppose that there is a discrete \mathbb{Z}_N isometry. Then, in a coordinate system adapted to this isometry, there is a vector k^m and a discrete Kaluza-Klein gauge field V_μ , such that the gauge transformations are $x^m \rightarrow x^m + \Lambda k^m$ and $V_\mu \rightarrow V_\mu - \partial_\mu \Lambda$, where Λ is a \mathbb{Z}_N -valued function of the coordinates x^m .

From the kinetic terms (2.16) and flux (2.8), the discrete gauge symmetries of the T^6/\mathbb{Z}_2 orientifold and its Calabi-Yau duals are

T^6/\mathbb{Z}_2 field	Charge	IIA CY ₃ field	Gauge symmetry
$c_{(2)4\mu}$	m	$b_{(2)5\mu}$	\mathbb{Z}_m (winding)
$c_{(2)5\mu}$	m	$b_{(2)4\mu}$	\mathbb{Z}_m (winding)
$b_{(2)4\mu}$	n	V^4_μ	\mathbb{Z}_n (isometry)
$b_{(2)5\mu}$	n	V^5_μ	\mathbb{Z}_n (isometry)

¹⁰I thank A. Dabholkar for emphasizing this point to me.

together with other discrete gauge symmetries that correspond to higher dimensional torsion cycles. We conclude that $H_1(X_6, \mathbb{Z}) = \mathbb{Z}_m \times \mathbb{Z}_m$ and that there is a $\mathbb{Z}_n \times \mathbb{Z}_n$ isometry in the Calabi-Yau threefold $X_6(m, n)$. This in turn tells us something about the fundamental group of X_6 , since $H_1(X_6, \mathbb{Z})$ is the abelianization of $\pi_1(X_6)$, defined as $\pi_1(X_6)$ modulo its commutator subgroup. In particular, the fundamental group is nontrivial for $m > 1$.

8. S-duality of T^6/\mathbb{Z}_2 as T-duality of abelian surface fibers

In section 2.2 we observed that interchange of the flux parameters (m, n) in the T^6/\mathbb{Z}_2 orientifold can be interpreted as S-duality followed by a 90° rotation in the 89-directions. On the other hand, in section 7 we found that there is a $\mathbb{Z}_m \times \mathbb{Z}_m$ winding symmetry and $\mathbb{Z}_n \times \mathbb{Z}_n$ isometry in the Calabi-Yau dual description. Since T-duality interchanges winding and isometry (NS B-field and metric), this suggests that $m \leftrightarrow n$ interchange can be interpreted as some type of T-duality of X_6 . Let us try to make this more precise.

In the T^6/\mathbb{Z}_2 orientifold, S-duality acts on the string coupling and string frame internal metric as¹¹

$$g_s \rightarrow \tilde{g}_s = 1/g_s, g_{mn}/\alpha' \rightarrow \tilde{g}_{mn}/\tilde{\alpha}' = g_{mn}/(g_s^2 \alpha'). \tag{8.1}$$

We can map these S-duality transformations to transformations of the Calabi-Yau Kähler moduli h and s . For h , we have

$$\frac{\bar{m}h}{(2\pi)^2 \alpha'_{\text{CY}}} = \frac{v'_1}{(2\pi)^2 \alpha'_{\text{CY}}} = \frac{v'_1 R'_9}{(2\pi/M_{11})^3} = \frac{v'_1 R'_9}{g'_s (2\pi\sqrt{\alpha'})^3} = \frac{1}{g_s}, \tag{8.2}$$

where primes denote quantities in the O6/D6 background of section 3.2, and the final $1/g_s$ on the right is in the T^6/\mathbb{Z}_2 orientifold. Therefore, the $m \leftrightarrow n$ interchange duality inverts the Kähler modulus h in string units:

$$\frac{\bar{m}h}{(2\pi)^2 \alpha'_{\text{CY}}} \rightarrow \frac{\tilde{\bar{m}}\tilde{h}}{(2\pi)^2 \tilde{\alpha}'_{\text{CY}}} = \frac{(2\pi)^2 \alpha'_{\text{CY}}}{\bar{m}h}, \tag{8.3}$$

where $(\tilde{m}, \tilde{n}) = (n, m)$. Since h gives the size of the abelian surface fiber (cf. eq. (6.7)), this makes precise the geometrical interpretation of the $m \leftrightarrow n$ duality. It is T-duality of the entire abelian surface fiber and relates two Calabi-Yau threefolds $X_6(m, n)$ and $X_6(n, m)$ of identical Hodge numbers.

The modulus s also gets rescaled in the $m \leftrightarrow n$ duality. This rescaling is governed by the equation

$$\left(\frac{2\tilde{s}}{(2\pi)^2 \tilde{\alpha}'_{\text{CY}}} \right) \left(\frac{\tilde{\bar{m}}\tilde{h}}{(2\pi)^2 \tilde{\alpha}'_{\text{CY}}} \right) = \left(\frac{2s}{(2\pi)^2 \alpha'_{\text{CY}}} \right) \left(\frac{\bar{m}h}{(2\pi)^2 \alpha'_{\text{CY}}} \right). \tag{8.4}$$

¹¹Here, $\alpha' = \tilde{\alpha}'$ is a common S-duality convention, but we will not need to specify a convention for the purposes of this section.

9. Lessons from (non)candidates for explicit Calabi-Yau constructions

The most thoroughly understood class of Calabi-Yau threefolds is the connected web of smooth Calabi-Yau threefolds that are hypersurfaces in 4D toric varieties. This class has been analyzed extensively by Kreuzer and Skarke, who tabulated all 473,800,776 reflexive polyhedra in four dimensions [12]. So, it is a logical first place to look for the Calabi-Yau duals of T^6/\mathbb{Z}_2 . We obtained a total of eight such manifolds $X_6(m, n)$:

M	$(h^{1,1}, h^{2,1})$	(m, n)
0	(2, 2)	(4, 1), (2, 2), (1, 4)
4	(6, 6)	(3, 1), (1, 3)
8	(10, 10)	(2, 1), (1, 2)
12	(14, 14)	(1, 1)

To review, M gives the number of divisors other than S and H , and (m, n) determines the group $\mathbb{Z}_m \times \mathbb{Z}_n$ of torsion 1-cycles and discrete isometries $\mathbb{Z}_n \times \mathbb{Z}_n$. The reduced pair $(\bar{m}, \bar{n}) = (m, n)/\text{gcd}(m, n)$ gives the polarization invariants of the abelian surface fibers and determines the intersection numbers of the threefold.

The hypersurface Calabi-Yau manifolds of Kreuzer and Skarke have trivial fundamental group aside from 16 exceptional cases, so we can at best expect to find the $m = 1$ duals of T^6/\mathbb{Z}_2 in this class.¹² Explicitly searching for the above Hodge numbers in the database of ref. [36], we find only the pair (14, 14), which appears three times. So, there are three candidate hypersurface Calabi-Yau threefolds. Each of these threefolds can be shown to be fibered by elliptic K3 surfaces in multiple ways [35].¹³ We do not have any reason to believe that our (14,14) abelian surface fibered threefold is also an elliptic K3 fibration, nevertheless this a logical possibility. On the other hand, there seems be a folk theorem that abelian surface fibered Calabi-Yau manifolds cannot be realized as toric hypersurfaces [37], so we deem this possibility unlikely.

Looking back at eq. (4.15), the appearance of the T^2 modular parameters τ_2 and τ suggests that we try to construct X_6 as free quotient of the product

$$A = E_{\tau_2} \times E_{\tau_2} \times E_{\tau} \tag{9.1}$$

of three elliptic curves, two of which are the same. In fact, Oguiso and Sakurai have considered an explicit construction of exactly this type.¹⁴ In ref. [39], these authors prove that the only Calabi-Yau threefolds that are given by quotients involving an abelian threefold A are of the form $X = A/G$, where G is a finite automorphism group acting freely on A , and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ or D_8 . They refer to (A, G) in these two cases as Igusa's pair and Igusa's refined pair, respectively. In each case, the resulting Calabi-Yau threefold is an abelian surface fibration over \mathbb{P}^1 . Moreover, in the first case $h^{1,1} = 3$ and in the second

¹²The exceptional cases were described in ref. [35] and can be ruled out explicitly. I thank M. Kreuzer for pointing out their existence and for subsequent email correspondence.

¹³I am grateful to B. Florea for insights on these threefolds and for introducing me to PALP [35].

¹⁴This example can only be realized as a complete intersection and not as a hypersurface [38], in agreement with the folk theorem mentioned in the previous paragraph.

case $h^{1,1} = 2$. So, the second case is a candidate for our three Calabi-Yau duals with Hodge numbers $(2, 2)$.

However, this candidate can be ruled out for two reasons. The simpler reason is that $H_1(X, \mathbb{Z})$ is not $\mathbb{Z}_m \times \mathbb{Z}_m$. From the explicit construction provided for Igusa's refined pair (Example (2.18) in ref. [39]), we can identify the generators of $\pi_1(X)$ explicitly and then abelianize to obtain $H_1(X, \mathbb{Z}) = (\mathbb{Z}_2)^6$. On more general grounds, we can employ the Cartan-Leray spectral sequence,¹⁵ to obtain the short exact sequence

$$0 \rightarrow H_1(G, \mathbb{Z}) \rightarrow H_1(A/G, \mathbb{Z}) \rightarrow H_1(A, \mathbb{Z})_G \rightarrow 0. \quad (9.2)$$

Here, $H_1(G, \mathbb{Z})$ is the abelianization of G , which for $G = D_8$ is $(\mathbb{Z}_2)^2$. The group $H^1(A, \mathbb{Z})_G$ is the \mathbb{Z}^6 lattice that defines the abelian threefold A , modulo identifications $x \cong gx$ for $g \in G$; in the case that $G = D_8$, one finds that this is $(\mathbb{Z}_2)^4$. Therefore, $H_1(A/G, \mathbb{Z}) = (\mathbb{Z}_2)^2 \times (\mathbb{Z}_2)^4$, in agreement with the explicit computation.

The second reason is potentially more useful. A Calabi-Yau manifold X is a quotient of an abelian threefold if and only if $c_2(X) = 0$. (See the discussion in ref. [39] and refs. [41, 42] contained therein.) For the purposes of section 6.4, we were content to show that $c_2(X_6) \cdot S = 0$. However, with a little more work, we can compute $c_2(X_6)$ explicitly. In the conventions of ref. [33], the precise statement of eqs. (6.11) and (6.12) is

$$\begin{aligned} F_1 &= -\frac{4\pi i}{12} \sum_{\alpha=1} (D_\alpha \cdot c_2(X_6)) t^\alpha + \text{worldsheet instantons}, \\ &= -\frac{4\pi i}{12} (M + 8) \tau_{\text{dil}} + \text{quantum corrections}, \end{aligned} \quad (9.3)$$

where the first line is the expression in the type IIA Calabi-Yau compactification and the second line is the expression in the T^6/\mathbb{Z}_2 orientifold with M D3 branes. The term proportional to M is due to the D3 branes and the term proportional to 8 is due to the O3 planes [43]. The normalization of the t_α is such that the shift symmetry is $t_\alpha \cong t_\alpha + 1$. Using the result (8.2), we have

$$\text{Im}(\tau_{\text{dil}}) = \bar{m} h / ((2\pi^2) \alpha'_{\text{CY}}) = \bar{m} t_H. \quad (9.4)$$

Therefore,

$$H \cdot c_2(X_6) = \bar{m} (M + 8) \quad \text{and} \quad D_\alpha \cdot c_2(X_6) = 0 \text{ otherwise.} \quad (9.5)$$

In particular, $c_2(X_6) \neq 0$, which means that

$$X_6 \text{ is not the quotient of an abelian threefold.} \quad (9.6)$$

This result has important implications for worldsheet instantons corrections in the type IIA compactification on X_6 , since it is expected that most (all?) Calabi-Yau manifolds that are not quotients of abelian threefolds contain rational curves [39]. Moreover, if such instantons exist in the Calabi-Yau duals, then there are corresponding worldsheet or D instanton corrections to the T^6/\mathbb{Z}_2 orientifold with $\mathcal{N} = 2$ flux. It would be very

¹⁵I thank V. Braun for explaining this and for directing me to section 8^{bis} of ref. [40].

interesting to identify explicit constructions of the Calabi-Yau duals and settle this issue. Either way we learn something new: the nonexistence of rational curves would be a counterexample to a conjecture that all Calabi-Yau manifolds other than quotients of abelian threefolds have rational curves; their existence would imply new corrections to the T^6/\mathbb{Z}_2 orientifold that have not been deduced by any other means.

10. Conclusions and outlook

Let us review the main results of this investigation. We have seen that the T^6/\mathbb{Z}_2 orientifold with $\mathcal{N} = 2$ flux has standard type IIA Calabi-Yau duals. Depending upon the choice of flux parameters (m, n) , we obtained eight possible dual Calabi-Yau threefolds $X_6(m, n)$, with Hodge numbers

$$(h^{1,1}, h^{2,1}) = (2, 2)^3, (6, 6)^2, (10, 10)^2, \text{ and } (14, 14)^1.$$

Here, the superscripts indicate degeneracies which are further distinguished by the integers (m, n) , satisfying

$$4mn = 16 - M, \quad \text{where} \quad h^{1,1} = h^{2,1} = M + 2.$$

The integers (m, n) determine the discrete gauge symmetries of the Calabi-Yau compactification,

$$\mathbb{Z}_m \times \mathbb{Z}_m \text{ winding, and } \mathbb{Z}_n \times \mathbb{Z}_n \text{ isometry,}$$

as well as the intersection numbers

$$H^2 \cdot S = 2\bar{m}\bar{n}, \quad H \cdot \mathcal{E}_I \cdot \mathcal{E}_J = -\bar{m}\delta_{IJ}, \quad \text{where} \quad (\bar{m}, \bar{n}) = (m, n)/\text{gcd}(m, n).$$

Finally, we have seen that the Calabi-Yau $X_6(m, n)$ is an abelian surface (T^4) fibration over \mathbb{P}^1 with polarization invariants (\bar{m}, \bar{n}) . We identified $m \leftrightarrow n$ interchange with S-duality times a 90° rotation in the T^6/\mathbb{Z}_2 orientifold, and with T-duality of the entire abelian surface fiber in the dual Calabi-Yau description.

We have not succeeded in providing explicit algebro-geometric constructions of these Calabi-Yau manifolds. However, by calculating $c_2(X_6)$ we were able to demonstrate that they are not free quotients of abelian threefolds (T^6). This makes it very likely that there exist rational curves in $X_6(m, n)$ and corresponding instantons in both the Calabi-Yau and T^6/\mathbb{Z}_2 descriptions. If this is so, then we have identified nonperturbative corrections in the T^6/\mathbb{Z}_2 orientifold that were not previously known. If it is not so, then we have identified Calabi-Yau manifolds that do not have rational curves and that are not quotients of abelian threefolds, which is also an interesting result.

By relating $\mathcal{N} = 2$, T^6/\mathbb{Z}_2 vacua to standard type IIA Calabi-Yau vacua, we have shown that the latter are part of a connected family in the sense of [14]. Without explicit constructions of the dual Calabi-Yau manifolds, there remains the stronger question of whether these vacua are connected to a large web through a common moduli space. Perturbative reasoning suggests that there are no extremal transitions. In the T^6/\mathbb{Z}_2 orientifold, there are only D3 branes. When two D3 branes meet, we do not perturbatively

expect to find a transition to a Higgs branch analogous to the transition that takes place when a D3 brane dissolves into a D7 brane and become worldvolume flux. However, this reasoning can fail nonperturbatively.¹⁶ An explicit construction of the Calabi-Yau duals would make it clear whether conifold transitions can occur.

For the future, we leave open the problem of explicitly constructing the Calabi-Yau duals of the T^6/\mathbb{Z}_2 based on the data provided here. We have seen that such duals always exist for T^6/\mathbb{Z}_2 with $\mathcal{N} = 2$ flux. A similar statement should hold for the $K3 \times T^2/\mathbb{Z}^2$ orientifold with $\mathcal{N} = 2$ flux.¹⁷ It would be interesting to perform the required dualities, which in this case amount to T-dualities in two of the K3 directions (i.e., mirror symmetry) and one of the T^2 directions. Finally, we would like to know whether there also exist heterotic duals in the $\mathcal{N} = 2$ case, and ultimately, to what extent flux compactifications have standard fluxless duals in the phenomenologically relevant case of $\mathcal{N} = 1$ supersymmetry.

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¹⁶I am indebted to S. Kachru for emphasizing this point.

¹⁷Indeed, this duality has subsequently been studied by P. Aspinwall [44].

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